4.4 Coordinate Systems

Example. Consider the standard basis in $\mathbb{R}^2$

$$B_s = \{\vec{e}_1, \vec{e}_2\} = \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\} \text{ and another basis}$$

$$B = \left\{ \underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{=\vec{b}_1}, \underbrace{\begin{bmatrix}2\\-1\end{bmatrix}}_{=\vec{b}_2} \right\} \text{ and } \vec{x} = \begin{bmatrix}8\\2\end{bmatrix}$$

Find c_1 and $c_2 \in \mathbb{R}$ such that

$$c_1 \begin{bmatrix}1\\1\end{bmatrix} + c_2 \begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}8\\2\end{bmatrix}. \quad \text{or} \quad A\vec{c} = \begin{bmatrix}8\\2\end{bmatrix} \text{ where } A = \begin{bmatrix}1 & 2\\1 & -1\end{bmatrix}$$

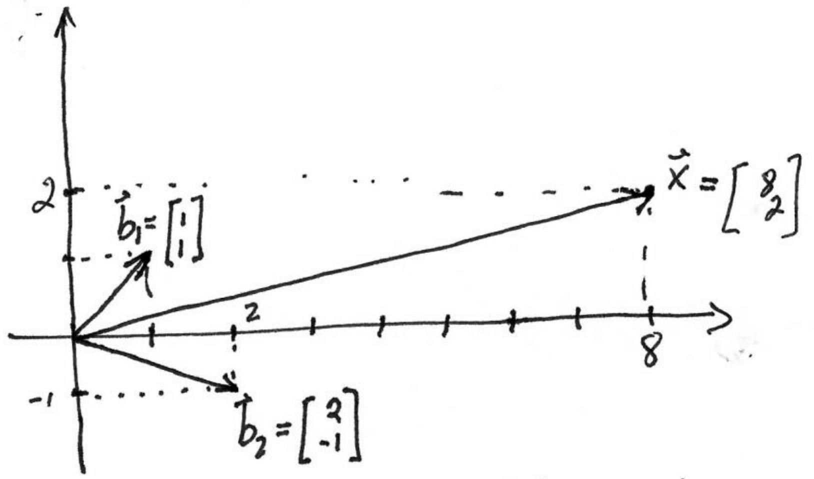
First, write augmented matrix and row-reduce it

$$\begin{bmatrix}1 & 2 & 8\\1 & -1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 8\\0 & -3 & -6\end{bmatrix} \Rightarrow \begin{array}{l} -3c_2 = -6 \Rightarrow \boxed{c_2 = 2} \\ c_1 = 8 - 2c_2 = 8 - 4 = 4 \\ \boxed{c_1 = 4} \end{array}$$

Thus,

$$4\begin{bmatrix}1\\1\end{bmatrix} + 2\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}8\\2\end{bmatrix}.$$

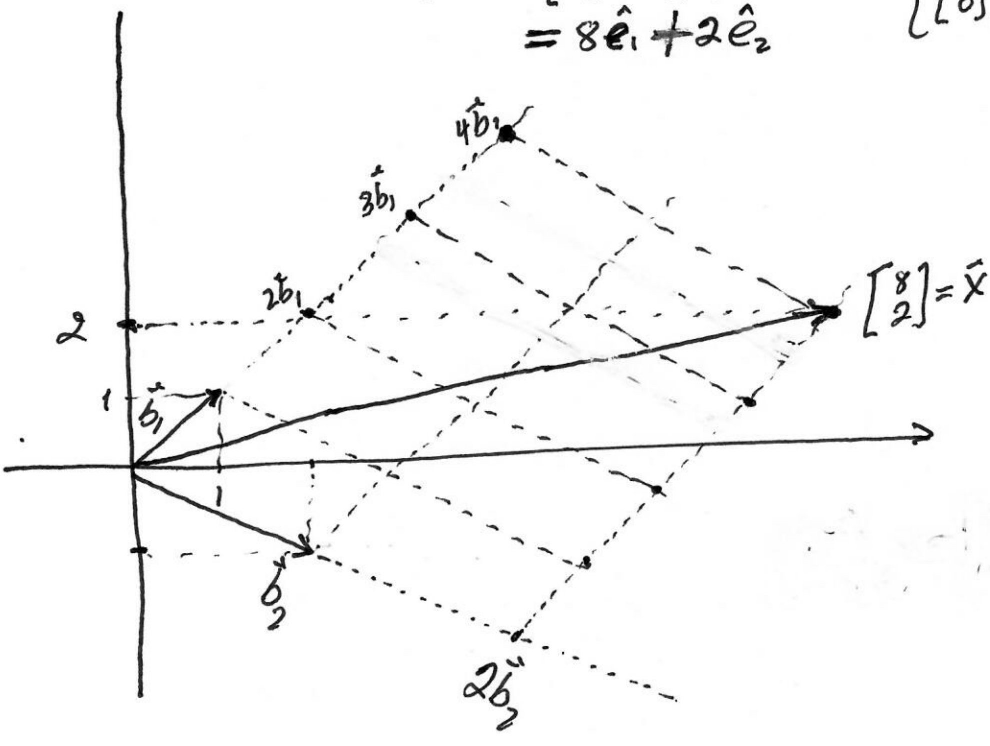
Geometrical representation



Representation with respect to standard basis $B_S = \{\hat{e}_1, \hat{e}_2\}$

$$\vec{x} = 8 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$= 8\hat{e}_1 + 2\hat{e}_2$$



Representation with respect to basis $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$

$$\vec{x} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 4\vec{b}_1 + 2\vec{b}_2$$

<u>Theorem 7.</u> $B = \{\vec{b}_1, .., \vec{b}_n\} \subset V$ (vector space) is a basis for V.

Then for each $\vec{x} \in V$, there exists a unique set of scalars $c_1, .., c_n$ such that

$$\vec{x} = c_1 \vec{b}_1 + \cdots + c_n \vec{b}_n.$$

<u>Proof.</u> Since $\vec{x} \in \text{Span}(B)$, then

$$\vec{x} = c_1 \vec{b}_1 + \cdots + c_n \vec{b}_n \qquad (3.1)$$

To show that the c's are unique, we assume

$$\vec{x} = d_1 \vec{b}_1 + \cdots + d_n \vec{b}_n \qquad (3.2)$$

and prove that $d_1 = c_1, .., d_n = c_n$.

In fact, subtracting (3.2) from (3.1)

$$\vec{0} = (c_1 - d_1)\vec{b}_1 + \cdots + (c_n - d_n)\vec{b}_n$$

$$\xRightarrow[\text{of } B]{\text{lin. indep.}} \quad \begin{array}{l} c_1 - d_1 = 0 \Rightarrow d_1 = c_1 \\ \vdots \\ c_n - d_n = 0 \Rightarrow d_n = c_n \quad \checkmark . \end{array}$$

<u>Def.</u> - For a basis $B = \{\vec{b}_1, .., \vec{b}_n\}$ of Vector space V and any Vector $\vec{x} \in V$, the coordinates of $\vec{x}$ relative to the basis B (or the B-coordinates of $\vec{x}$) are the coefficients $c_1, .., c_n$ such that

$$\vec{x} = c_1 \vec{b}_1 + \cdots + c_n \vec{b}_n .$$

Notation

$$[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

In our previous example,

$$B = \left\{ \overset{\vec{b}_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \overset{\vec{b}_2}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}} \right\}, \quad \vec{x} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}, \quad \vec{x} = 4\vec{b}_1 + 2\vec{b}_2$$

then, $$[\vec{x}]_B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

If we call $P_B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$, then

$$\vec{x} = \begin{bmatrix} 8 \\ 2 \end{bmatrix} = 4\vec{b}_1 + 2\vec{b}_2 = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = P_B [\vec{x}]_B$$

or $$\boxed{\vec{x} = P_B [\vec{x}]_B}.$$

The matrix P_B is called the <u>change of coordinates matrix</u>. It transforms a vector in $\mathbb{R}^n$ $[\vec{x}]_b$ represented with respect to the basis B into the same vector represented with resp. to standard basis.

In $\mathbb{R}^n$, if

$B = \{\vec{b}_1, .., \vec{b}_n\}$ is a basis for $\mathbb{R}^n$, $\vec{x} \in \mathbb{R}^n$

and

$$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \dots + c_n\vec{b}_n$$

then,

$$[\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \quad P_B = [\vec{b}_1 \dots \vec{b}_n]$$

and

$$\vec{x} = P_B\, [\vec{x}]_B \qquad (5.1)$$

This means that if we know the coordinates of $\vec{x}$ relative to a basis B different than the standard basis, then we can obtain the coordinates of $\vec{x}$ relative to the standard basis using (5.1)

The typical situation is to <u>find the coordinates of $\vec{x}$ relative to a basis B (not stand. basis) given.</u>

Knowing that P_B is invertible, because its column vectors are linearly independent (Basis of $\mathbb{R}^n$), then from (5.1)

$$\boxed{[\vec{x}]_B = P_B^{-1}\,\vec{x}.} \qquad (6.1).$$

Therefore, to find the coordinates of $\vec{x}$ relative to a new basis B, we just need to apply (6.1).

In our previous example,

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\} \quad \text{and} \quad \vec{x} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

$$P_B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad P_B^{-1} = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix}$$

Therefore $$[\vec{x}]_B = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 12/3 \\ 6/3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

or $$[\vec{x}]_B = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \text{ for } \vec{x} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

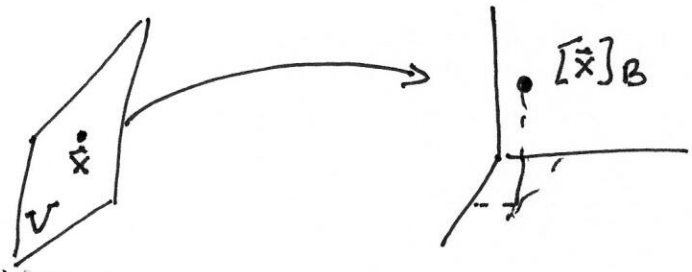
Coordinate Mapping

Consider a Vector Space V

and a basis $B = \{\vec{b}_1, .., \vec{b}_n\}$ of V

We define the mapping

$$T : V \longrightarrow \mathbb{R}^n$$



Vector Space

Polynomials degree $\leq n-1$

Matrices $m \times m$

Definition of T: For every $\vec{x} \in V$,

$$\vec{x} = c_1\vec{b}_1 + \cdot\cdot + c_n\vec{b}_n, \text{ then } T(\vec{x}) = [\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

<u>thm 8.-</u> $\mathbb{V}$ is a vector space

$B = \{\vec{b}_1, \dots, \vec{b}_n\}$ is a basis of $\mathbb{V}$.

Then, the <u>coordinate mapping</u>

$$T: \mathbb{V} \longrightarrow \mathbb{R}^n$$
$$\vec{x} \longrightarrow [\vec{x}]_B \quad \text{is a linear transformation}$$

and it is <u>one-to-one</u> and <u>onto</u> $\mathbb{R}^n$.

<u>Proof.-</u> It's linear because

If $\vec{u}, \vec{v} \in \mathbb{V}$ are such that $\vec{u} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$, $\vec{v} = d_1\vec{b}_1 + \dots + d_n\vec{b}_n$

then $[\vec{u} + \vec{v}]_B = T(\vec{u} + \vec{v}) =$

$$= T\left((c_1+d_1)\vec{b}_1 + \dots + (c_n+b_n)\vec{b}_n\right) =$$

$$= \begin{bmatrix} c_1+d_1 \\ c_2+d_2 \\ \vdots \\ c_n+d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = [\vec{u}]_B + [\vec{v}]_B$$

Similarly

$$[r\vec{u}]_B = T(r\vec{u}) = T\left(rc_1\vec{b}_1 + \dots + rc_n\vec{b}_n\right) =$$

$$= \begin{bmatrix} rc_1 \\ \vdots \\ rc_n \end{bmatrix} = r\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = r\,[\vec{u}]_B.$$

Proof that is one-to-one and onto in the exercises 23, 24.

<u>Definition</u> V, W vector spaces

$T: V \longrightarrow W$ linear transformation ①

If T is one-to-one ② and onto W ③

T is called <u>an isomorphism between $\mathbf{V}$ and W.</u>

It's said that V and W are indistinguishable as vector spaces.

"Same person with different dress."

<u>Example 5 (book)</u>

$B = \{1, t, t^2, t^3\}$ standard basis of $\mathbb{P}_3$

So $\vec{p}$ in $\mathbb{P}_3$ can be written as

$$\vec{p} = p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

then
$$[\vec{p}]_B = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$T: \mathbb{P}_3 \longrightarrow \mathbb{R}^4$$

$$\vec{p} = a_0 + a_1 t + \cdots + a_3 t^3 \longrightarrow [\vec{p}]_B = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Isomorphism from $\mathbb{P}_3$ to $\mathbb{R}^4$.

Explain its meaning.

Sect 4.4

#29) Consider $S = \{ \overset{\vec{p}_1}{(1-t)^2}, \overset{\vec{p}_2}{t-2t^2+t^3}, \overset{\vec{p}_3}{(1-t)^3} \}$

Use coordinate vectors to test the linear independence of S.

$(1-t)^2 = 1 - 2t + t^2$

$(1-t)^3 = (1-2t+t^2)(1-t) = 1 - 2t + t^2 - t + 2t^2 - t^3$

or $\vec{p}_3 = 1 - 3t + 3t^2 - t^3 \Rightarrow [\vec{p}_3]_S = \begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix}$

$\vec{p}_2 = t - 2t^2 + t^3 \Rightarrow [p_2]_S = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$

And $\vec{p}_1 = 1 - 2t + t^2 \Rightarrow [p_1]_S = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

To check if $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ is a linearly independent set is enough to show whether or not the set $\{[\vec{p}_1]_S, [\vec{p}_2]_S, [\vec{p}_3]_S\}$

$= \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix} \right\}$ is linearly indep.

Verification:

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -3 \\ 1 & -2 & 3 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The vectors are linearly dependent.
Not every column is a pivot column.

$c_1 = -c_3$

$c_2 = c_3$

c_3 free

If $c_3 = 1$, $c_2 = 1$, $c_1 = -1$

$$(-1)\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$